UDC 539. 374

ON THE THEOREM OF UNIQUENESS FOR TWO - MODE ELASTOPLASTIC PROCESSES

PMM Vol.42, № 2, 1978, pp. 377 - 383

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(Received August 13, 1976)

The quasistatic boundary value problem for two-mode elastoplastic processes and the problem of determining the initial velocity of the points of a body after the break for an arbitrary elastoplastic process with the break following the simple deformation, are both formulated. It is shown for elastoplastic processes with a corner point, that under certain assumptions about the physical functions describing the material properties, the variation in the external load determines the initial velocities of the particles of the body in a unique manner, and therefore also the corner angle in the strain trajectory at each point. The theorem of uniqueness of the solution of the boundary value problem is proved for the loads which induce the two-mode elastoplastic processes where regions of active deformations as well as the regions of unloading may appear in the body.

1. **Analysis of the stress-strain relations.** In the case of a two-mode elastoplastic process with the strain trajectory developing a corner of angle θ $(0 < \theta < \pi)$ at the instant t, the relations between the stress σ_{ij} and strain ε_{ij} tensors have the form

$$S_{ij} = \frac{2}{3}\frac{\Phi(\partial_u)}{\partial_u}\partial_{ij}, \quad t \leqslant t_0 \tag{1.1}$$

$$S_{ij} = \frac{2}{3}\frac{\sigma_u}{\sin\theta}[p_{ij}\sin(\theta - \vartheta) + p_{ij}^\circ \sin\vartheta], \quad t > t_0$$

$$\sigma = 3K\varepsilon \tag{1.2}$$

where

$$\sigma = {}^1/_3\sigma_{ii}, \quad S_{ij} = \sigma_{ij} - \sigma\delta_{ij}, \quad \sigma_u = ({}^3/_2 S_{ij}S_{ij})^{1/2}$$

$$\varepsilon = {}^1/_3\varepsilon_{ii}, \quad \partial_{ij} = \varepsilon_{ij} - \varepsilon\delta_{ij}, \quad \partial_u = ({}^2/_3\partial_{ij}\partial_{ij})^{1/2}$$

$$p_{ij} = (\partial_{ij} - \partial_{ij}^\circ)/(s - s^\circ), \quad p_{ij}^\circ = \partial_{ij}^\circ/\partial_u^\circ, \quad \partial_{ij}^\circ = \partial_{ij}(t_0)$$

$$s = \begin{cases} \partial_u, & t \leqslant t_0 \\ s^\circ - s^\circ\cos\theta + \sqrt{\partial_u^2 - (s^\circ\sin\theta)^2}\,\operatorname{sign}\cos\theta, & t > t_0 \end{cases}$$

$$\vartheta = \arccos\left(\mathbf{p}\frac{\boldsymbol{\sigma}}{\sigma_u}\right) = \arccos\left(p_{ij}\frac{S_{ij}}{\sigma_u}\right)$$

$$s^\circ \equiv \partial_u(t_0) = \partial_u^\circ, \quad s^\circ \in [\varepsilon_s, \lambda]$$

Here s is the arc length of the strain trajectory, s° the length of the simple deformation segment (ε_s denotes the yield point and λ is a quantity of the order of several ε_s), p_{ij}° is the direction tensor on the simple deformation segment, $\Phi(s)$ denotes the stress intensity during the simple deformation, p_{ij} with $s > s^\circ$ is the direction tensor of the strain trajectory past the corner, K is the bulk elasticity modulus and ϑ

is the angle of convergence i.e. the angle between the stress vector σ and the vector $\mathbf{p}$ of the tangent to the strain trajectory, corresponding to the tensor p_{ij} [3].

In accordance with the isotropy postulate [4], we have $\sigma_u = \sigma_u(s^\circ, \theta; s)$, $\vartheta = \vartheta(s^\circ, \theta; s)$. The relations (1.1) hold for any value of the corner angle θ, but the material functions σ_u and ϑ for the directions of the active deformation and unloading are different. It is usually assumed [1,4] that in the unloading region the stress and strain increments are linearly connected. When the deformation anisotropy is neglected, we have

$$S_{ij} - S_{ij}^\circ = 2G(\partial_{ij} - \partial_{ij}^\circ) \tag{1.3}$$

where G denotes the shear modulus.

We adopt the following assumptions for σ_u and ϑ which are in agreement with the experimental data:

1°. $\sigma_u \in C^1(s^\circ, s^\circ + h)$ for any values of s° and θ, and $\sigma_u(s^\circ, \theta; s^\circ) = \Phi(s^\circ)$; $\sigma_u(s^\circ, \theta; s) \to \Phi(s)$ as $\theta \to 0$, $s > s^\circ$; there exists $\sigma_u'(s^\circ + 0, \theta) = \lim \partial\sigma_u(s^\circ, \theta; s)/\partial s$ as $s \to s^\circ$, $s > s^\circ$.

Experiments have shown [5,6] that the quantity $\partial\sigma_u(s^\circ, \theta; s)/\partial s$ is finite for any $s \in (s^\circ, s^\circ + h)$, $\theta \in (0, \pi/2]$. Here h is the trace of the delay of the vector properties. For small $\Delta s = s - s^\circ$ and large $\theta \in (0, \pi/2]$ more accurate experiments are needed. The experimental data available admit two possibilities: $\sigma_u'(s^\circ + 0, \pi/2) = 0$ (experiments in [5,6]) or $\sigma_u'(s^\circ + 0, \theta) < 0$ (experiments in [7,8]). For the work hardening materials the following approximation [9] can be used:

$$\sigma_u'(s^\circ + 0, \theta) = \Phi'(s^\circ)\cos\theta \tag{1.4}$$

where $\Phi'(s)$ is the hardening modulus during a simple deformation, and according to [7] we have the approximation

$$\sigma_u'(s^\circ + 0, \theta) = \Phi'(s^\circ) - B\theta^n \tag{1.5}$$

where B and n are material constants.

2°. $\vartheta \in C^1(s^\circ, s^\circ + h)$ for any s° and θ; $\vartheta(s^\circ, \theta; s) = \theta$ for $s = s^\circ$; $\vartheta(s^\circ, \theta; s) \to 0$ as $\theta \to 0$ for any $s > s^\circ$; $-\infty < M < \partial\vartheta(s^\circ, \theta; s)/\partial s < 0$ for $s \in (s^\circ, s^\circ + h)$; a $\vartheta'(s^\circ + 0, \theta) = \lim \partial\vartheta(s^\circ, \theta; s)/\partial s$ exists for $s \to s^\circ$, $s > s^\circ$, $\vartheta(s^\circ, \theta; s) \equiv 0$ when $s > s^\circ + h$ for any $s^\circ \in [\varepsilon_s, \lambda]$ and $\theta \in (0, \pi/2]$.

In particular, the following approximation is used in [7]:

$$\vartheta'(s^\circ + 0, \theta) = -C\theta/\Phi(s^\circ) \tag{1.6}$$

where C is a material constant.

3°. The function $U = \sigma_u \cos\vartheta$ increases is s when s° and θ are fixed (see the experiments in [6], and (*)).

When $\theta \to 0$ and $s > s^\circ$, the second relations of (1.1) yield the relations of the theory of small elastoplastic deformations [1]. The following limit exists under the assumptions 1° and 2°:

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*) Lenskii V.S. Study of plasticity of metals under complex loading. Doctoral dissertation, MGU, 1960)

$$S_{ij}^{\cdot +} = \lim_{\substack{t \to t_0 \\ t > t_0}} \frac{S_{ij}(t) - S_{ij}(t_0)}{t - t_0} = 2G^1(s^\circ, \theta) V_{ij} + 2G^\circ(S^\circ, \theta) v_u p_{ij}{}^\circ \tag{1.7}$$

where
$$V_{ij} = \frac{d\partial_{ij}}{dt}(s^\circ, \theta;\ s^\circ + 0) \qquad G^1(s^\circ, \theta) = -\frac{\Phi(s^\circ)\, \vartheta'(s^\circ + 0, \theta)}{3 \sin\theta} \tag{1.8}$$

$$G^\circ(s^\circ, \theta) = {}^1/_3\, [\sigma_u'(s^\circ + 0, \theta) + \Phi(s^\circ)\, \vartheta'(s^\circ + 0, \theta) \operatorname{ctg}\theta] \tag{1.9}$$

where V_{ij} is the deviator of the strain rate tensor at the instant t_0, and v_u is the corresponding intensity.

Formulas (1.3), (1.7) and (1.9) imply that $G^1(s^\circ, \theta) \equiv G$, $G^\circ(s^\circ, \theta) \equiv 0$ in the region of unloading, i.e. when $\theta \in [\pi/2, \pi]$, therefore we have

$$\sigma_u'(s^\circ + 0, \theta) = 3G\cos\theta \qquad \vartheta'(s^\circ + 0, \theta) = -3G\sin\theta/\Phi(s^\circ) \tag{1.10}$$

Assuming now that the function $S_{ij}^{\cdot +}$ is continuous in θ, we obtain $\sigma_u'(s^\circ + 0, \pi/2) = 0$.

2. Classes of functions used. Let D be a set of continuously differentiable vector functions $\mathbf{u}$ defined on a bounded region $\Omega \subset R^3$ and satisfying [10] the conditions

$$\int_\Omega \mathbf{u}\, dx = 0, \quad \int_\Omega [\mathbf{u} \times \mathbf{r}]\, dx = 0 \tag{2.1}$$

The following Korn inequality [10] holds in the domain D:

$$I_1 \leqslant C_1 I_\varepsilon \qquad I_1 = \int_\Omega \frac{\partial \mathbf{u}}{\partial x_i} \frac{\partial \mathbf{u}}{\partial x_i}\, dx, \quad I_\varepsilon = \int_\Omega \varepsilon_{ij}(\mathbf{u})\, \varepsilon_{ij}(\mathbf{u})\, dx \tag{2.2}$$

Here C_1 is a constant depending on Ω only. The Poincaré inequality [11]

$$I_2 \leqslant C_2 I_1 + C_3 \left| \int_\Omega \mathbf{u}\, dx \right|^2 \qquad I_2 = \int_\Omega |\mathbf{u}|^2\, dx \tag{2.3}$$

holds for the functions belonging to D. Here C_2 and C_3 are constants depending on Ω only. Therefore, by virtue of (2.2) and (2.3) we find, that the Sobolev norms $|\mathbf{u}|_{(1)}^2 = I_1$, $\|\mathbf{u}\|_{(1)}^2 = I_1 + I_2$ and the norm [12]

$$\|\mathbf{u}\|^2 = I_\varepsilon \tag{2.4}$$

are equivalent.

Let $H(\Omega)$ denote a Hilbert space obtained by the closure of D in the norm (2.4).

3. Uniqueness theorem. In the case of two-mode elastoplastic processes the problem of equilibrium of a body occupying the region Ω with the boundary Γ can be stated in the form of the following boundary value problem: it is required to determine the vector displacement function $\mathbf{u}$, the deformation tensor ε_{ij} and the stress tensor σ_{ij} which satisfy, in the region Ω, the following requirement for every $t \in [0, T]$

$$\mathbf{u} \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma)$$
$$\varepsilon_{ij}, \sigma_{ij} \in C^1(\Omega) \cap C^\circ(\Omega \cup \Gamma) \qquad (i, j = 1, 2, 3)$$

as well as the relations (1.1) and (1.2), the Cauchy relations

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

equations of equilibrium and the boundary conditions

$$\frac{\partial \sigma_{ij}}{\partial x_j} + F_i = 0 \quad \text{in } \Omega \tag{3.1}$$

$$u_i = \psi_i \text{ on } \Gamma_u \quad \sigma_{ij}\nu_j = T_{\nu i} \text{ on } \Gamma_\sigma \ (\Gamma_u \cup \Gamma_\sigma = \Gamma). \tag{3.2}$$

The mass forces $\mathbf{F}$, the stresses $\mathbf{T}_\nu$ and the displacement $\boldsymbol{\psi}$ at the boundary are all given functions. Let us assume that $\mathbf{F}, \mathbf{T}_\nu$ and $\boldsymbol{\psi}$ vary in such a manner that a simple deformation trajectory is realized [1, 13] on the interval $[0, t_0]$ at any point of the body. Let $\mathbf{u}^0$ be the displacement vector at the instant t_0, corresponding to the solution of the problem of simple deformation [14].

Differentiating the relations (3.1) and (3.2) with respect to t $(t > t_0)$, performing the passage to the limit $t \to t_0 + 0$ and taking into account the formulas (1.7) – (1.9) and (1.2), we obtain the following system of differential equations for determining the initial velocity of the particles beyond the corner point.

$$G^1(s^\circ, \theta)\left(\Delta V_i - \frac{\partial}{\partial x_i}\operatorname{div}\mathbf{V}\right) + \frac{\partial}{\partial x_j}(2G^1(s^\circ, \theta))V_{ij} + \tag{3.3}$$

$$K\frac{\partial}{\partial x_i}\operatorname{div}\mathbf{V} + G^\circ(s^\circ, \theta)\frac{v_u}{\partial_u^\circ}\left(\Delta u_i^\circ - \frac{\partial}{\partial x_i}\operatorname{div}\mathbf{u}^\circ\right) +$$

$$\frac{\partial}{\partial x_j}\left(2G^\circ(s^\circ, \theta)\frac{v_u}{\partial_u^\circ}\right)\partial_{ij}^\circ + F_i^{\cdot +} = 0$$

$$V_i = \psi_i^{\cdot +} \quad \text{on } \Gamma_u, \quad \sigma_{ij}^{\cdot +}\nu_j = T_{\nu i}^{\cdot +} \quad \text{on } \Gamma_\sigma \tag{3.4}$$

where

$$F_i^{\cdot +} = \lim \frac{dF_i}{dt}, \quad T_{\nu i}^{\cdot +} = \lim \frac{dT_{\nu i}}{dt}, \quad \psi_i^{\cdot +} = \lim \frac{d\psi_i}{dt}$$
$$(t \to t_0), \quad (t > t_0)$$

We shall consider, for the sake of simplicity, the second boundary value problem in which the boundary Γ is acted upon by the surface forces $\mathbf{T}_\nu$ only, i.e. when $\Gamma_\sigma = \Gamma$, $\Gamma_u = 0$. Multiplying scalarly the system of equations (3.3) by any vector function continuously differentiable in Ω and integrating over the whole volume we obtain, with (3.4) taken into account, an integral identity. We shall call the vector function $\mathbf{V} \in H(\Omega)$ satisfying this integral identity for any continuously differentiable vector function $\boldsymbol{\varphi}$, a generalized solution of the boundary value problem (3.3), (3.4).

Let us reduce the boundary value problem (3.3), (3.4) to the operator form. The assumptions that the functions $G^1(s^\circ, \theta)$ and $G^\circ(s^\circ, \theta)$ are continuous in $(s^\circ, \theta) \in [\varepsilon_s, \lambda] \times [0, \pi]$ imply, that they are bounded. In this case we can separate a linear bounded functional in $\boldsymbol{\varphi} \in H(\Omega)$ from the integral identity in question for any $\mathbf{V} \in H(\Omega)$. The Riesz theorem [15] will then enable us to find the operator $a(\mathbf{V})$ acting from $H(\Omega)$ into $H(\Omega)$. We have

$$(a(\mathbf{V}), \boldsymbol{\varphi}) \equiv \int_\Omega [2G^1(s^\circ, \theta)V_{ij} + 2G^\circ(s^\circ, \theta)v_u p_{ij}^\circ + \tag{3.5}$$

$$3K\delta_{ij}\operatorname{div}\mathbf{V}]\,\varepsilon_{ij}(\boldsymbol{\varphi})\,dx=\int_{\Omega}F_i^{\cdot+}\varphi_i dx+\int_{\Gamma}T_{\nu i}^{\cdot+}\varphi_i dx,\quad \forall\boldsymbol{\varphi}\in H(\Omega)$$

We shall call the operator $a(\mathbf{V})$ the fundamental operator of the boundary value problem (3.3),(3.4) and assume that the integrals in the right hand side of (3.5) define linear bounded integrals in the space $H(\Omega)$. Reference [16] gives various conditions under which the necessary requirement is realized, e.g. $F_i\in L_p(\Omega)$ $(\forall p\geqslant {}^6/_5)$ and $T_{\nu i}^{\cdot+}\in L_q(\Gamma)$ $(\forall q\geqslant {}^4/_3)$. According to the Riesz theorem there exists $\mathbf{f}\in H(\Omega)$ such that

$$(\mathbf{f},\boldsymbol{\varphi})=\int_{\Omega}F_i^{\cdot+}\varphi_i dx+\int_{\Gamma}T_{\nu i}^{\cdot+}\varphi_i dx,\quad \forall\boldsymbol{\varphi}\in H(\Omega)$$

Let us find the sufficient conditions which must be satisfied by the functions $\sigma_u'(s^\circ+0,\theta)$ and $\vartheta'(s^\circ+0,\theta)$ for the operator $a(\mathbf{V})$ to be strictly monotonous on the set $H(\Omega)$ [17], i.e. for the inequality

$$(a(\mathbf{V}^1)-a(\mathbf{V}^2),\ \mathbf{V}^1-\mathbf{V}^2)\geqslant 0 \tag{3.6}$$

to hold for any $\mathbf{V}^1,\ \mathbf{V}^2\in H(\Omega)$ and to become an equality only when $\mathbf{V}^1=\mathbf{V}^2$ in $H(\Omega)$.

Let $\mathbf{v}$ and $\mathbf{w}$ be the vectors in a five-dimensional space $\mathcal{E}^5$ [3] corresponding to the deviators V_{ij} and $\mathcal{E}_{ij}(\boldsymbol{\varphi})$. We set

$$A(\mathbf{v})=-\frac{\Phi(s^\circ)\,\vartheta'(s^\circ+0,\theta)}{\sin\theta}\,\mathbf{v}+[\sigma_u'(s^\circ+0,\theta)+\Phi(s^\circ)\operatorname{ctg}\theta\,\vartheta'(s^\circ+0,\theta)]\,v_u\mathbf{p}^\circ \tag{3.7}$$

Taking into account (3.5), (1.7) —(1.9) and (3.7) we obtain

$$(a(\mathbf{V}),\boldsymbol{\varphi})=\int_{\Omega}A(\mathbf{v})\,\mathbf{w}dx+\int_{\Omega}K\operatorname{div}\mathbf{V}\operatorname{div}\boldsymbol{\varphi}dx \tag{3.8}$$

Since $\mathbf{p}^\circ$ is a known vector belonging to the five-dimensional space, any vector $\mathbf{v}$ admits the representation

$$\mathbf{v}=\mathbf{v}_u(\mathbf{p}^\circ\cos\theta+\mathbf{n}\sin\theta),\ \theta\in[0,\pi] \tag{3.9}$$

where $\mathbf{n}$ is the unit vector orthogonal to $\mathbf{p}^\circ$ and belongs to the plane $(\mathbf{v},\ \mathbf{p}^\circ)$.

Using (3.7) and (3.9), we obtain the following relation for any $\mathbf{V}^1,\mathbf{V}^2\in D$:

$$\begin{aligned}[A(\mathbf{v}^1)-A(\bar{\mathbf{v}}^2)]\,[\mathbf{v}^1-\mathbf{v}^2]=\{\Phi(s^\circ)\,[-\mathbf{n}^1\vartheta'(s^\circ+0,\theta^1)v_u^1+\mathbf{n}^2\vartheta'\times\\(s^\circ+0,\theta)\,v_u^2]+\mathbf{p}^\circ[\sigma_u'(s^\circ+0,\theta^1)\,v_u^1-\sigma_u'(s^\circ+0,\theta^2)\,v_u^2]\}\times\\ [\mathbf{p}^\circ(v_u^1\cos\theta^1-v_u^2\cos\theta^2)+\mathbf{n}^1\sin\theta^1 v_u^1-\mathbf{n}^2\sin\theta^2 v_u^2]\end{aligned} \tag{3.10}$$

The indices 1 and 2 in (3.10) refer to the values of the functions corresponding to the vectors $\mathbf{V}^1$ and $\mathbf{V}^2$. Since $-1\leqslant\mathbf{n}^1\,\mathbf{n}^2\leqslant 1$ and $\vartheta'(s^\circ+0,\theta)<0$, on the basis of (3.8) and (3.10) we arrive at the following inequality:

$$\begin{aligned}&(a(\mathbf{V}^1)-a(\mathbf{V}^2),\mathbf{V}^1-\mathbf{V}^2)\geqslant\\&\int_{\Omega}\{[\sigma_u'(s^\circ+0,\theta^1)\,v_u^1-\sigma_u'(s^\circ+0,\theta^2)\,v_u^2]\,(v_u^1\cos\theta^1-v_u^2\cos\theta^2)+\\&\Phi(s^\circ)\,[-\vartheta'(s^\circ+0,\theta^1)\,v_u^1+\vartheta'(s^\circ+0,\theta^2)\,v_u^2]\,(v_u^1\sin\theta^1-\\&v_u^2\sin\theta^2)\}\,dx+K\int_{\Omega}(\operatorname{div}\mathbf{V}^1-\operatorname{div}\mathbf{V}^2)^2\,dx\end{aligned} \tag{3.11}$$

Using the Cartesian coordinates $x = v_u \sin\theta$, $y = v_u \cos\theta$ for $\theta \in [0, \pi]$, $v_u \in R$, the integrand function of the first integral in the right-hand side of (3. 11) assumes the form

$$(f_1(x_1, y_1) - f_1(x_2, y_2))(x_1 - x_2) + (f_2(x_1, y_1) - f_2(x_2, y_2))(y_1 - y_2) \tag{3.12}$$

where

$$f_1(\theta, v_u) = -\Phi(s^\circ)\,\vartheta'(s^\circ + 0, \theta)\, v_u \quad f_2(\theta, v_u) = \sigma_u'(s^\circ + 0, \theta)\, v_u \tag{3.13}$$

Let us rewrite the expression (3. 12) with help of the Lagrange theorem, as follows:

$$(x_1 - x_2, y_1 - y_2)\begin{Vmatrix} \frac{\partial f_1}{\partial x}(x^*, y_2) & \frac{\partial f_1}{\partial y}(x_1, y^*) \\ \frac{\partial f_2}{\partial x}(x^{**}, y_2) & \frac{\partial f_2}{\partial y}(x_1 y^{**})) \end{Vmatrix}\begin{Vmatrix} x_1 - x_2 \\ y_1 - y_2 \end{Vmatrix} \tag{3.14}$$

where x^* and x^{**} fall between x_1 and x_2 and y^*, y^{**} fall between y_1 and y_2. The necessary and sufficient conditions for the quadratic form

$$\sum_{i,j=1}^{2} a_{ij}\zeta_i\zeta_j$$

where a_{ij} are the components of the matrix appearing in (3. 4) to be positive definite is, that

$$\frac{\partial f_1}{\partial x} > 0, \quad \frac{\partial f_2}{\partial y} > 0 \tag{3.15}$$

$$\frac{\partial f_1}{\partial x}(z_1)\frac{\partial f_2}{\partial y}(z_2) - \frac{1}{4}\left[\frac{\partial f_1}{\partial y}(z_3) + \frac{\partial f_2}{\partial x}(z_4)\right]^2 > 0$$

Here z_i $(i = 1, 2, 3, 4)$ are arbitrary points in the half-plane $x \geqslant 0$ of the variables (x, y).

Let us assume that the functions $\sigma_u'(s^\circ + 0, \theta)$ and $\vartheta'(s^\circ + 0, \theta)$ belong to the class $C^1(0, \pi)$ $\forall\, s^\circ \in [\varepsilon_s, \lambda]$ and satisfy the conditions (3. 15). Then the right-hand side of the inequality (3. 11) is nonnegative and the inequality (3. 6) holds. On the other hand, if $(a(\mathbf{V}^1) - a(\mathbf{V}^2), \mathbf{V}^1 - \mathbf{V}^2) = 0$, then by virtue of (3. 8) –(3. 11) $V_{ij}^1 = V_{ij}^2$, $\operatorname{div} \mathbf{V}^1 = \operatorname{div} \mathbf{V}^2$ and the vector functions $\mathbf{V}^1$ and $\mathbf{V}^2$ are equal to each other as elements of $H(\Omega)$, i. e. almost everywhere in Ω.

The inequality (3. 6) remains valid for any $\mathbf{V}^1, \mathbf{V}^2 \in H(\Omega)$ provided that the functions $\sigma_u'(s^\circ + 0, \theta)$, $\vartheta'(s^\circ + 0, \theta)$ are piecewise continuously differentiable in $\theta \in [0, \pi]$ and satisfy the conditions (3. 15).

Introducing the notation

$$g_1(s^\circ, \theta) \equiv -\Phi(s^\circ)\left[\frac{\partial\vartheta'(s^\circ + 0, \theta)}{\partial\theta}\cos\theta + \vartheta'(s^\circ + 0, \theta)\sin\theta\right]$$

$$g_2(s^\circ, \theta) \equiv -\frac{\partial\sigma_u'(s^\circ + 0, \theta)}{\partial\theta}\sin\theta + \sigma_u'(s^\circ + 0, \theta)\cos\theta$$

$$g_3(s^\circ, \theta) \equiv \Phi(s^\circ)\left[\frac{\partial\vartheta'(s^\circ + 0, \theta)}{\partial\theta}\sin\theta - \vartheta'(s^\circ + 0, \theta)\cos\theta\right]$$

$$g_4(s^\circ, \theta) \equiv \frac{\partial\sigma_u'(s^\circ + 0, \theta)}{\partial\theta}\cos\theta + \sigma_u'(s^\circ + 0, \theta)\sin\theta$$

we can write the conditions (3. 15) with (3. 13) taken into account, in the form

$$g_1(s^\circ, \theta) > 0, \quad g_2(s^\circ, \theta) > 0$$

$$\min_{0\leqslant\theta\leqslant\pi} g_1(s^\circ, \theta) \min_{0\leqslant\theta\leqslant\pi} g_2(s^\circ, \theta) > \frac{1}{4} \max_{0\leqslant\theta,\, \theta^*\leqslant\pi} [g_3(s^\circ, \theta) + g_4(s^\circ, \theta^*)]^2 \qquad (3.16)$$

The fact established above which states that the fundamental operator $a(\mathbf{V})$ of the boundary value problem (3.3), (3.4) is strictly monotonous, leads to the following uniqueness theorem.

Theorem 1. Let the external loads $\mathbf{F}^{\cdot+}$ and $\mathbf{T}_\nu^{\cdot+}$ be such, that the integrals in the right-hand side of (3.5) define linear bounded functionals in the space $H(\Omega)$, and the formulas (3.16) hold. Then the problem (3.3), (3.4) cannot have more than one solution.

Thus the variation in the external loads determines uniquely the initial velocity of the particles and hence the corner angles on the deformation trajectory at all points of the body, since

$$\cos\theta = \frac{2}{3} \frac{V_{ij}}{v_u} p_{ij}^\circ$$

Theorem 1 is related to the local characteristics of the process and can consequently be applied to any elastoplastic process with a corner point appearing after a simple deformation.

Theorem 2. We assume that the external loads vary with time in the interval $[0, T]$, so that a deformation process represented by a two-segment broken line occurs at every point of the body. Under the conditions of Theorem 1, the solution of the boundary value problem (3.1), (3.2) for two-mode elastoplastic processes is unique when the stress vector is given on the whole of the boundary.

Proof. Let us suppose that $\mathbf{u}^1, \mathbf{u}^2 \in H(\Omega)$ are the solutions of the boundary value problem (3.1), (3.2), with ε_{ij}^1 and ε_{ij}^2 denoting the corresponding deformation tensors. According to Theorem 1, for given $F_i^{\cdot+}$ and $T_{\nu i}^{\cdot+}$ the corner angle θ and the initial value of the deformation rate tensor $\mathbf{V}$ beyond the corner point on the deformation trajectory can be determined uniquely, i.e. $\theta^1 = \theta^2$ and $p_{ij}^1 = p_{ij}^2$ almost everywhere in Ω. We shall prove that $s^1 = s^2$ and $\operatorname{div}\mathbf{u}^1 = \operatorname{div}\mathbf{u}^2$ almost everywhere in Ω.

Every solution $\mathbf{u}^1$ and $\mathbf{u}^2$ satisfies the relation of the form

$$\int_\Omega [S_{ij}\partial_{ij}(\varphi) + K \operatorname{div}\mathbf{u} \operatorname{div}\varphi]\, dx = \int_\Omega F_i\varphi_i dx + \int_\Gamma T_{\nu i}\varphi_i dx \qquad (3.17)$$

for any vector function $\varphi \in H(\Omega)$. In particular, when $\varphi = \mathbf{u}^1 - \mathbf{u}^2$ we can use (3.17) to obtain the following equation for $\mathbf{u}^1$ and $\mathbf{u}^2$:

$$\int_\Omega (S_{ij}^1 - S_{ij}^2)(\partial_{ij}^1 - \partial_{ij}^2)\, dx + K \int_\Omega (\operatorname{div}\mathbf{u}^1 - \operatorname{div}\mathbf{u}^2)^2 dx = 0 \qquad (3.18)$$

Since for two-mode elastoplastic processes we have

$$(S_{ij}^1 - S_{ij}^2)(\partial_{ij}^1 - \partial_{ij}^2) = (\sigma_u^1 \cos\theta^1 - \sigma_u^2 \cos\theta^2) \times (s^1 - s^2)$$

we arrive, taking into account the assumption 3°, at the conclusion that $(S_{ij}^1 - S_{ij}^2)$ $(\vartheta_{ij}^1 - \vartheta_{ij}^2) \geqslant 0$. Consequently the relation (3.18) holds if and only if $s^1 = s^2$ and $\operatorname{div} \mathbf{u}^1 = \operatorname{div} \mathbf{u}^2$ almost everywhere in Ω. Thus, $\varepsilon_{ij}^1 = \varepsilon_{ij}^2$ almost everywhere in Ω, i.e. $\mathbf{u}^1 = \mathbf{u}^2$ in $H(\Omega)$ and almost everywhere in Ω, and this completes the proof of Theorem 2.

Note. Here t represents a parameter used to discriminate a sequence of events. This parameter varies monotonously with the length of the deformation trajectory.

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Translated by L.K.